Leader Election in SINR Model with Arbitrary Power Control

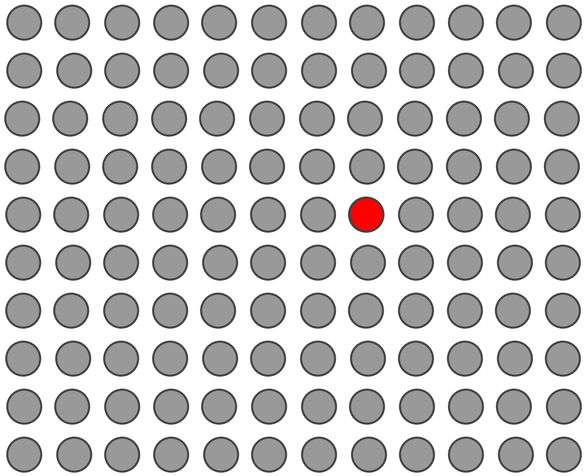
Magnús M. Halldórsson, Stephan Holzer,
Evangelia Anna Markatou

Leader election can be done really fast.

We just need to shout (very) loudly.

The problem: Leader Election in SINR

Given *n* nodes in a wireless network, pick one.



Our solution:

The nodes shout: “***I am the leader***”

The loudest one wins....

Multiple Access Channel Model

Given *n* nodes in a network:

Everyone hears: if one node broadcasts.

No one hears: if (i) no one broadcasts or
(ii) more than one nodes broadcast

Leader election: O(log n)

Signal to Interference plus Noise Ratio (SINR) model

Interference adds up.

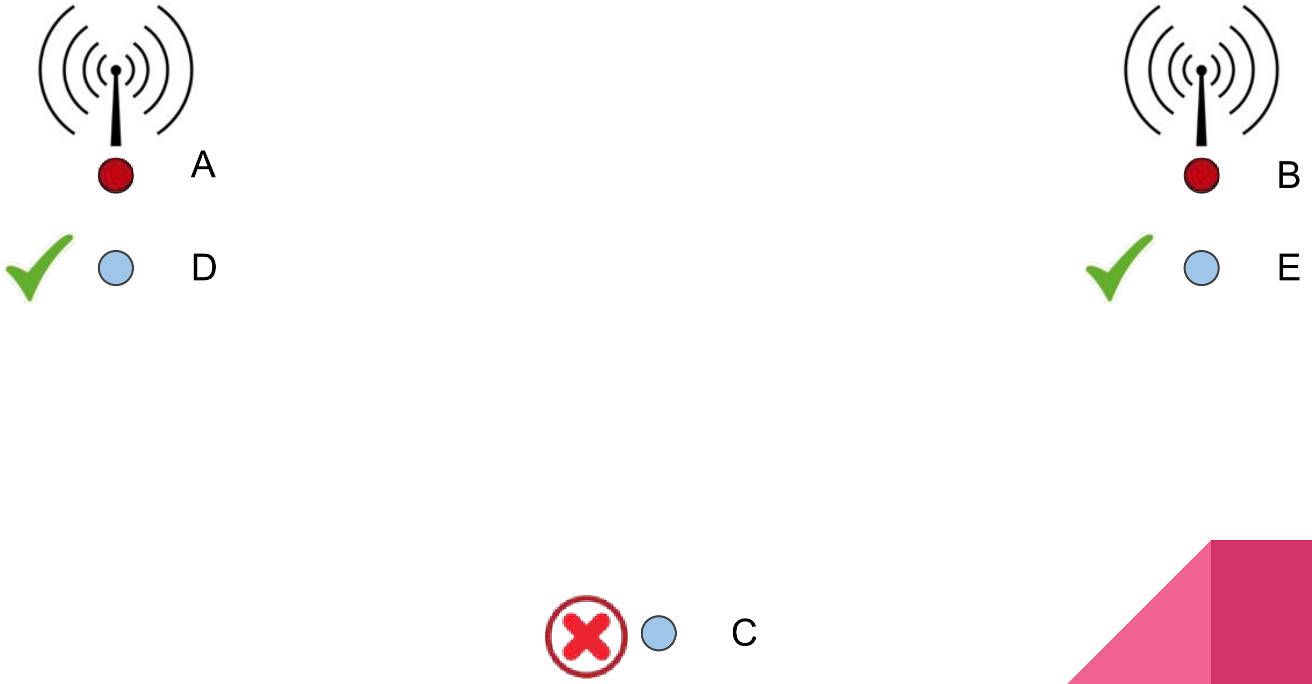
More capabilities:

(i) Capture Effect

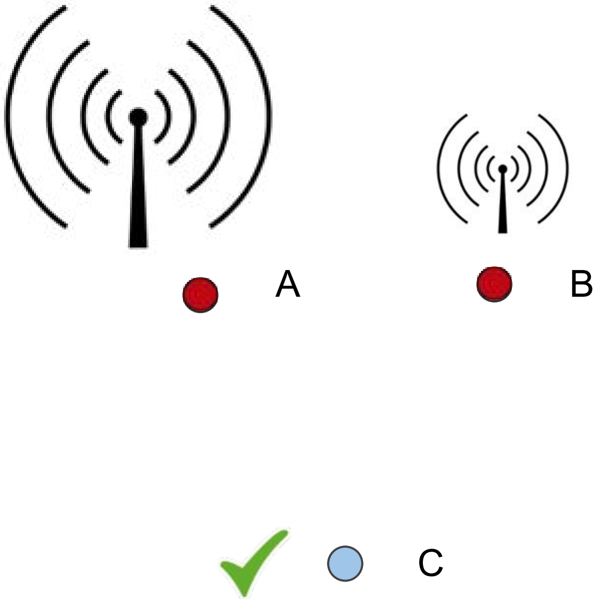
(ii) Power Control

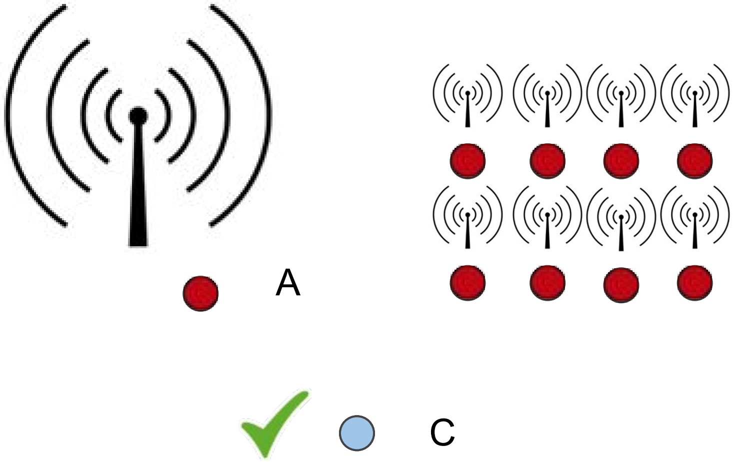
Leader election: O(log n) with uniform power

The Magic of the Capture Effect



The Power of Power Control





SINR: More formally

- We have n nodes in a single-hop wireless network.
- Time is divided into synchronous rounds.
- On each round, a node can either broadcast or listen.
- A node v can receive a message transmitted by node u , iff v is listening and

$$SINR(u, v, I) = \frac{\frac{P_u}{d(u,v)^\alpha}}{N + \sum_{w \in I} \frac{P_w}{d(w,v)^\alpha}} \geq \beta,$$

- R is the ratio of the longest to shortest distance between any two nodes in the network, and is bounded by a polynomial in n

2-Round Leader Election Protocol

- Shout loudly enough!
- Pick a power from a big enough range
- Approximate *n* using a geometric random variable
- Acknowledge leader loudly enough

Actual protocol

Variables:

- Role: competitor OR listener
- k: Geometric(½) random variable
- ID: Pick from $[2^k k^4, 2 \cdot 2^k k^4]$
- P: Transmission power

$$P = P_{min} \cdot ID^{\gamma ID}$$

Communication Rounds:

Round 1:
if $Role_v$ = competitor **then**
  Broadcast ID_v using power P_v
else
  Receive $Leader_v$
Round 2:
if $Role_v$ = competitor **then**
  Receive $Leader_v$
else
  Broadcast $Leader_v$ using power P_v

Analysis

Make sure:

1. Exactly one node holds the maximum ID.

2. If that node broadcasts, and there is at least one listener, the listener will receive its message

Analysis: The leader is unique

Lemma 1. *Let $k_1 := \log n - \log\log n - 2$. For at least one and at most $8\log n$ competitors v does it hold that $k_v \geq k_1$, with probability greater than $1 - \frac{1}{8n}$.*

Let A_v be the event that a given node v is a competitor and has $k_v \geq k_1$.

$$\frac{2\log n}{n} = 2^{-1-k_1} \leq \Pr[A_v] \leq 2^{-k_1} = \frac{4\log n}{n}$$

The probability that no node satisfies A_v is then at most

$$\left(1 - \frac{2\log n}{n}\right)^n \leq e^{-2\log n} \leq \frac{1}{16n}$$

Analysis: The leader is unique

Lemma 1. *Let $k_1 := \log n - \log\log n - 2$. For at least one and at most $8 \log n$ competitors v does it hold that $k_v \geq k_1$, with probability greater than $1 - \frac{1}{8n}$.*

Let A_v be the event that a given node v is a competitor and has $k_v \geq k_1$.

$$\frac{2\log n}{n} = 2^{-1-k_1} \leq \Pr[A_v] \leq 2^{-k_1} = \frac{4\log n}{n}$$

Let X be the number of nodes v for which A_v holds.
Then $2 \log n \leq E[X] \leq 4 \log n$ and by Chernoff bound:

$$\Pr[X \geq 8\log n] \leq \Pr[X \geq 2[X]] \leq 2^{-0.55[X]} < 2^{-2.2\log n} \leq \frac{1}{16n}$$

Analysis: The leader is unique

Lemma 2. *A sole competitor receives the highest ID with probability greater than* $1 - \frac{1}{8n}$*, given that at least one node calculated* $k_v \geq k_1$*.*

The ranges of IDs assigned to nodes of different k values are disjoint.

The range from which the IDs are chosen is $[J, 2J]$, for $J \geq 2^{k_1} k_1^4 \geq \frac{n \cdot \log^3 n}{8}$.

The probability that some pair of nodes with the highest k value pick the same ID is at most

$$\frac{(8 \log n)^2}{\frac{n \cdot \log^3 n}{8}} = \frac{512}{n \log n} < \frac{1}{8n}$$

Analysis: The listener receives a message

Lemma 3. *If a sole competitor receives the highest ID, then its transmission is received by all the listeners.*

Let w be the sole competitor with the highest ID, and u be a listener.

$$\frac{\frac{P_w}{d(u,w)^\alpha}}{N + \sum_{v \in I} \frac{P_v}{d(u,v)^\alpha}}$$

Analysis: The listener receives a message

Lemma 3. *If a sole competitor receives the highest ID, then its transmission is received by all the listeners.*

Let w be the sole competitor with the highest ID, and u be a listener.

$$\frac{\frac{P_w}{d(u,w)^\alpha}}{N + \sum_{v \in I} \frac{P_v}{d(u,v)^\alpha}} \geq \frac{\frac{P_w}{d(u,w)^\alpha}}{N + n\frac{P_v}{d(u,v)^\alpha}}$$

Let P_v be the second highest transmission power, and $d(u,v)$ be the smallest distance between any node and u.

Analysis: The listener receives a message

Lemma 3. *If a sole competitor receives the highest ID, then its transmission is received by all the listeners.*

Let w be the sole competitor with the highest ID, and u be a listener.

$$\frac{\frac{P_w}{d(u,w)^\alpha}}{N + \sum_{v \in I} \frac{P_v}{d(u,v)^\alpha}} \geq \frac{\frac{P_w}{d(u,w)^\alpha}}{(n+\beta)\frac{P_v}{d(u,v)^\alpha}}$$

Let's bound N.

$$N \leq \frac{P_v}{d(u,v)^\alpha \cdot \beta}$$

Analysis: The listener receives a message

Lemma 3. *If a sole competitor receives the highest ID, then its transmission is received by all the listeners.*

Let w be the sole competitor with the highest ID, and u be a listener.

$$\frac{\frac{P_w}{d(u,w)^\alpha}}{N + \sum_{v \in I} \frac{P_v}{d(u,v)^\alpha}} \geq \frac{P_w}{(n+\beta)n^{c\alpha}P_v}$$

What about $\frac{d(u,w)}{d(u,v)}$?

$$\frac{d(u,w)}{d(u,v)} \leq R \leq n^c$$

Analysis: The listener receives a message

Lemma 3. *If a sole competitor receives the highest ID, then its transmission is received by all the listeners.*

Let w be the sole competitor with the highest ID, and u be a listener.

$$\frac{\frac{P_w}{d(u,w)^\alpha}}{N + \sum_{v \in I} \frac{P_v}{d(u,v)^\alpha}} \geq \frac{P_w}{(n+\beta)n^{c\alpha} P_v}$$

What about $\frac{P_w}{P_v}$? $P = P_{min} \cdot ID^{\gamma ID}$

$$\frac{P_w}{P_v} = \frac{P(ID_w)}{P(ID_v)} \geq \frac{P(ID_w)}{P(ID_w - 1)}$$

Analysis: The listener receives a message

Lemma 3. *If a sole competitor receives the highest ID, then its transmission is received by all the listeners.*

Let w be the sole competitor with the highest ID, and u be a listener.

$$\frac{\frac{P_w}{d(u,w)^\alpha}}{N + \sum_{v \in I} \frac{P_v}{d(u,v)^\alpha}} \geq \frac{n^\gamma}{(n+\beta)n^{c\alpha}}$$

What about $\frac{P_w}{P_v}$? $\quad P = P_{min} \cdot ID^{\gamma ID}$

$$\frac{P_w}{P_v} \geq ID_w^\gamma \geq n^\gamma$$

Analysis: The listener receives a message

Lemma 3. *If a sole competitor receives the highest ID, then its transmission is received by all the listeners.*

Let w be the sole competitor with the highest ID, and u be a listener.

$$\frac{\frac{P_w}{d(u,w)^\alpha}}{N + \sum_{v \in I} \frac{P_v}{d(u,v)^\alpha}} \geq \frac{n^\gamma}{(n+\beta)n^{c\alpha}} \geq \beta$$

Lower bound on the Power Range

Theorem 4. *Every 2-round leader election algorithm in the SINR model running correctly w.h.p. requires a power range* $2^{\Omega(n)}$.

Consider n nodes located in a unit metric.

The winner must be heard by a listener in the first round.

We shall calculate the probability that no two nodes use the highest transmission power subrange, and correlate that to the power range needed.

Lower bound on the Power Range

Theorem 4. *Every 2-round leader election algorithm in the SINR model running correctly w.h.p. requires a power range* $2^{\Omega(n)}$.

We divide the available range of power into subranges, each within factor 2.

Let p_i be the probability that node v transmits from a power in the ith subrange.

Let q be the largest number such that: $$\sum_{i=1}^{q} p_i \leq \frac{1}{2n}$$

Lower bound on the Power Range

Theorem 4. *Every 2-round leader election algorithm in the SINR model running correctly w.h.p. requires a power range* $2^{\Omega(n)}$.

We divide the available range of power into subranges, each within factor 2.

Let A_i be the event that at least two nodes use the i-th highest subrange, B_i be the event that no node transmits at higher subranges. Let $C_i = A_i \cap B_i$.

$$\Pr[C_i] = \Pr[A_i \cap B_i] = \Pr[A_i | B_i] \Pr[B_i]$$

But, $\Pr[A_i | B_i] \geq \Pr[A_i]$

Thus, $\Pr[C_i] \geq \Pr[A_i] \Pr[B_i]$

Lower bound on the Power Range

Theorem 4. *Every 2-round leader election algorithm in the SINR model running correctly w.h.p. requires a power range* $2^{\Omega(n)}$.

Let's calculate the probability of A_i, that at least two nodes use the i-th highest subrange

$$\Pr[A_i] > \binom{n}{2} p_i^2 (1 - p_i)^{n-2} > \frac{n^2}{3} p_i^2 \left(1 - \frac{1}{2n}\right)^{n-2} > \frac{n^2}{3e} p_i^2$$

Let's calculate the probability of B_i , the event that no node transmits at higher subranges

$$\Pr[B_i] \geq 1 - n \sum_{j=1}^{i-1} p_i \geq \frac{1}{2}$$

Lower bound on the Power Range

Theorem 4. *Every 2-round leader election algorithm in the SINR model running correctly w.h.p. requires a power range* $2^{\Omega(n)}$.

Let C be the union of all C_i. Note that all C_i are mutually exclusive, and use the Cauchy-Schwarz inequality:

$$\Pr[C] \geq \sum_{i=1}^{q} \Pr[C_i] \geq \frac{n^2}{3e} \sum_{i=1}^{q} p_i^2 \cdot \frac{1}{2} \geq \frac{n^2}{6e} \frac{\left(\sum_{i=1}^{q} p_i\right)^2}{q} \geq \frac{1}{24e \cdot q}$$

Our algorithm fails when C holds, thus if $\Pr[C] \leq 1/n$

then $q \geq n/(24e) = \Omega(n)$

Multiple round protocol

Repeat the 2-round protocol t times.

Use a slower growing ID function: $g_t(k) = 2^k k^{3t+1}$

and a slower growing power function:

$$P = P_{min} \cdot ID_v^{\gamma(ID_v)^{1/t}}$$

After each round the listeners update their leader to the largest value heard so far.

Multiple round protocol - Analysis

Similar to the 2-round algorithm

Make sure that each 2-round run of the original algorithm works with probability at least

$$1 - \frac{1}{n^{1/t}}$$

Then, the algorithm works with probability at least

$$1 - \left(\frac{1}{n^{1/t}}\right)^t$$

Thank you!

This work is presented at PODC and SIROCCO, where it received the best paper award.